AIR FORCE INSTITUTE OF TECHNOLOGY

AIR UNIVERSITY
UNITED STATES AIR FORCE

SCHOOL OF ENGINEERING

WRIGHT–PATTERSON AIR FORCE BASE, OHIO

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RECURSION FORMULAE TO OBTAIN INTEGRAL ROOTS OF REAL NUMBERS

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By
Eugene M. Romer
Captain, USAF

Assistant Professor of Aeronautical Engineering

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Abstract

Generalized recursion sequences to obtain the integral root of a number are developed. These sequences require, at most, the use of the square root operation in addition to the basic arithmetic operations; consequently, the sequences are suitable for use with desk calculators or manual computations. For ~~If~~ the $n^{\underline{th}}$ root of a number, where n is an intérger in the range

$$2^{k-1} < n \leq 2^{k}, \quad (k = \text{integer}),$$

the development in this paper provides for k + 1 recursion sequences, all of which exhibit second order convergence as the required root is approached.

INTRODUCTION

The problem of obtaining a recursion sequence to obtain any integral root of a number using a desk calculator or hand computation has been examined in the literature and a number of special forms of a more general recursion sequence have been derived.[1,2] In this paper, the most general recursion sequence is obtained, and the criteria for the selection of the appropriate parameters in the recursion sequences are developed.

Although the procedure here is for the determination of a positive integral root, this method is the key to determination of a fractional root or a negative integral root. In the former case, where the root to be obtained is of the form, p/q, one merely obtains the q^{th} root by the methods in this paper and raises the answer to the power, p. The determination of a negative integral root of a number, N, is the same as finding the positive integral root of the reciprocal of N.

Analysis:

In order to solve the problem, $L = A^{1/s}$, where A and s are given, recursion sequences of the form given in equation (1) are suggested in references 1 and 2:

$$A_{n+1} = \gamma A_n + \beta \left(A_n^h A\right)^{\frac{1}{s+h}} \qquad , (1)$$

where $s = 2^k - h$, $1 \leq h \leq 2^{k-1}$,

(h and k are positive integers), and γ and β are constants to be chosen so as to make the convergence as rapid as possible. The form shown above can be generalized to:

$$A_{n+1} = \gamma A_n + \beta (A_n^r A)^{\frac{2^m}{s+h}} \quad , (2)$$

where s, h, γ, and β have the same meaning as in equation (1), and m and r are integers which are to be determined. If we define the fractional error ϵ_n , of the n^{th} estimate to the desired root by the equation

$$A_n = L(1+\epsilon_n) \quad , (3)$$

and noting that $L^s = A$, we may use (3) in (2) to obtain

$$1+\epsilon_{n+1} = \gamma(1+\epsilon_n) + \beta(1+\epsilon_n)^{\left(\frac{r}{s+h}\right)2^m} (L)^{\left\{\left(\frac{r+s}{s+h}\right)2^m - 1\right\}} \quad , (4)$$

If we require that the resulting error equation be independent of L or A, then the exponent of the term involving L must equal zero, i.e., we require that

$$\left(\frac{r+s}{s+h}\right)2^m = 1 \quad , (5)$$

and under this condition, equation (4) becomes

$$1+\epsilon_{n+1} = \gamma(1+\epsilon_n) + \beta(1+\epsilon_n)^{\left(\frac{r}{s+h}\right)2^m} \quad , (6)$$

Since $s+h = 2^k$, we obtain from (5):

$$r = 2^{k-m} - s \quad , (7)$$

From (7), r will be an integer as long as $m \leq k$.

We will require that $\epsilon_{n+1} = 0$ when $\epsilon_n = 0$,

which when used in equation (6) gives:

$$\gamma + \beta = 1 \quad , (8)$$

From (6), we also note that $\epsilon_{n+1} = -1$, when $\epsilon_n = -1$, i.e., an initial estimate greater than zero is always required. The slope of ϵ_{n+1} vs ϵ_n is given by

$$\frac{\partial \epsilon_{n+1}}{\partial \epsilon_n} = \gamma + \beta\left(\frac{r}{s+h}\right)2^m\left(1+\epsilon_n\right)^{-(2^{m-k})s} \quad , (9)$$

For convergence to the point, $\epsilon_{n+1} = \epsilon_n = 0$, it is sufficient to require that

$$\left| \frac{\partial \epsilon_{n+1}}{\partial \epsilon_n} \right| < 1 \quad , (10)$$

Using this criterion applied at the origin, i.e., from (9) and (10), when $\epsilon_n = 0$, we obtain

$$\left| \gamma + \beta\left(\frac{r}{s+h}\right)2^m \right| < 1 \quad , (11)$$

or using (7) and (8) in (11):

$$\left| 1 - \beta\left(\frac{s}{s+h}\right)2^m \right| < 1 \quad , (12)$$

or

$$\left| s(1-2^m) + h + s\gamma 2^m \right| < s+h$$

The most rapidly converging trajectories in the vicinity of the origin will be those whose slope = 0 at $\epsilon_n = 0$, i.e., those whose trajectories satisfy (using equation (9)):

$$\gamma + \beta\left(\frac{r}{s+h}\right)2^m = 0 \quad , (14)$$

The simultaneous solution of (8) and (14) gives:

$$\beta = \frac{1}{2^m}\left(1+\frac{h}{s}\right) \qquad , (15a)$$

$$\gamma = 1-\frac{1}{2^m}\left(1+\frac{h}{s}\right) \qquad , (15b)$$

Equations of the form given by equation (2) will <u>always</u> be convergent as long as $\left|\frac{\partial \epsilon_{n+1}}{\partial \epsilon_n}\right|$ for large ϵ_n is less or equal to 1.

From (9), this requires then $\gamma \leq 1$ or

$$\frac{1}{2^m}\left(1+\frac{h}{s}\right) \geq 0 \qquad , (16)$$

which imposes no real restriction. Since for large ϵ_n, $\frac{\partial \epsilon_{n+1}}{\partial \epsilon_n} \approx \gamma$, i.e.,

$$\frac{\partial \epsilon_{n+1}}{\partial \epsilon_n} \approx 1 - \frac{2}{s}^{k-m} \quad , \text{ for } \epsilon_n >> 1 \qquad , (17)$$

Consequently, the larger the value m, the slower the rate of convergence at the larger values of ϵ_n. On the other hand, increasing the value of m reduces the effort required to obtain subsequent estimates. For m= k-1, only one square root in each cycle must be computed; for m = k, each cycle requires no operations other than the four basic arithmetic operations. From equation (17), $\frac{\partial \epsilon_{n+1}}{\partial \epsilon_n} > 0$ for large ϵ_n, for integer values of m $\geq$ 1. For the case where m = 0, $\frac{\partial \epsilon_{n+1}}{\partial \epsilon_n} < 0$, for large ϵ_n and there exists the possibility of ϵ_{n+1} becoming negative for sufficiently large positive values of ϵ_n.

The range of ϵ_n for which the m= 0 recursion sequence will converge can be obtained by setting m = 0 and ϵ_{n+1} = -1 in (6) and solving for ϵ_n.

The range of ϵ_n for which the m = 0 sequence converges is

$$-1 < \epsilon_n < \epsilon_n^* \quad , (18)$$

where ϵ_n^* is given by

$$\epsilon_n^* = \left(1 + \frac{s}{h}\right)^{(1+\frac{h}{s})} - 1 \quad , (19)$$

Note that as long as $0 < h < s$, then

$$\left(1+\frac{s}{h}\right)^{(1+h/s)} - 1 > (1 + s/h) - 1 = s/h$$

i.e., $\epsilon_n^* > s/h$ which establishes a range of ϵ_n from -1 to s/h. In this situation, however, the discussion of range of convergence is purely academic insofar as practical application is concerned; the divergence is readily recognized by the appearance of a negative value of A_{n+1}, and can be eliminated by the simple expedient of taking some fractional value of A_n for the value of A_{n+1}. The appearance of a positive subsequent estimate generates a solution.

The use of recursion formulae of the form suggested above, (for m=k), yields particularly simple results for some of more common integral roots. In particular, the formula for the square root of A is given by

$$A_{n+1} = \frac{1}{2}\left(A_n + \frac{A}{A_n}\right) \quad , (20)$$

which is both simple and rapidly convergent.

Since useful recursion sequences may be obtained for each integer value of m from o to k, it follows that one has a choice of k + 1 sequences of the form in (2) for any given problem.

References

1. Taylor, G. R., "An Approximation For Any Positive Integral Root," Mathematics Magazine, Vol. 35, March, 1962. p. 107-8.

2. Romer, E. M., "An Iterative Procedure for Obtaining Fractional Roots of Real Numbers, Wright Air Development Center, WADC TN 59-116, Wright-Patterson AFB, Ohio, April, 1959.